Matrices Having the Interval Property

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Abstract

We say that a class $\mathcal{C}$ of n-by-n matrices possesses the *interval property* if for any n-by-n interval matrix $[A] = ([\underline{a}_{ij}, \overline{a}_{ij}])_{i,j=1,\ldots,n}$ the membership $[A] \subset \mathcal{C}$ can be inferred from the membership to $\mathcal{C}$ of a specified set of its vertex matrices; here a *vertex matrix* of $[A]$ is a real matrix $B = (b_{ij})_{i,j=1,\ldots,n}$ with $b_{ij} \in \left\{\underline{a}_{ij}, \overline{a}_{ij}\right\}$ for all $i, j = 1, \ldots, n$. Examples of such classes include the

- M-matrices or, more generally, inverse-nonnegative matrices [12], where only the bound matrices $\underline{A}$ and $\overline{A}$ are required to be in the class;
- inverse M-matrices [11], where all vertex matrices are needed;
- positive definite matrices [5], [15], where a subset of cardinality 2^{n-1} is required (here only symmetric matrices in $[A]$ are considered).

A class of matrices which in the nonsingular case are somewhat related to the inverse nonnegative matrices are the totally nonnegative matrices. A real matrix is called *totally nonnegative* if all its minors are nonnegative. Such matrices arise in a variety of ways in mathematics and its applications, e.g., in differential and integral equations, numerical mathematics, combinatorics, statistics, and computer aided geometric design. For background information we refer to the monographs [6], [14]. The speaker posed in 1982 the conjecture that the set of the nonsingular totally nonnegative matrices possesses the interval property, where only two vertex matrices are involved [7], see also [6, Section 3.2], [14, Section 3.2]. The two vertex matrices are the bound matrices with respect to the *checkerboard ordering* which is obtained from

the usual entry-wise ordering in the set of the square matrices of fixed order by reversing the inequality sign for each entry in a checkerboard fashion. The Cauchon algorithm [2] (also called deleting derivation algorithm [10] and Cauchon reduction algorithm [13]) was used in [1] to settle the conjecture. Also, a fixed zero-nonzero pattern of the minors stays unchanged through an interval of nonsingular totally nonnegative matrices [1].

As a generalization of the totally nonnegative matrices we further consider *sign regular matrices*, i.e., matrices with the property that all their minors of fixed order have one specified sign or are allowed also to vanish. We identify some subclasses of the sign regular matrices which exhibit the interval property. The subclasses which require to check only two vertex matrices include the following sets (here it is understood that the two bound matrices have the same signature of their minors) [3]:

- the strictly sign regular matrices, i.e., the matrices with the property that all their minors of fixed order have one (strict) specified sign;
- the nonsingular almost strictly sign regular matrices, a class in between the nonsingular sign regular matrices and the strictly sign regular matrices;
- the tridiagonal nonsingular sign regular matrices;
- the nonsingular totally nonpositive matrices, i.e., the matrices with the property that all their minors are nonpositive.

In some instances, the assumption of nonsingularity can be somewhat relaxed [1], [3], [4]. These results lead us to the following new conjecture: Assume that the two bound matrices with respect to the checkerboard ordering are nonsingular and sign regular; then all matrices lying between the two bound matrices are nonsingular and sign regular, too. It was shown in [8] that the conclusion is true if we consider instead of the two bound matrices a set of vertex matrices with the cardinality of at most 2^{2n-1} (n being the order of the matrices).

We present further classes of matrices having the interval property but requiring in general more than two vertex matrices, see [9]. These include the nonsingular matrices, P-matrices, diagonal stable matrices, Hurwitz and Schur stable matrices.

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